Linear Algebra 1

10/10/2025

1 Linear equations (10 + 15 = 25 pts)

Three people play a game in which there are always two winners and one looser. They have the understanding that the loser gives each winner an amount equal to what the winner already has. After three games, each has lost just once and each has €24.

(a) Let x_k be the amount of money Player k begins with. Assume that Player 1 has lost the first game, Player 2 the second game, and Player 3 the third game. Write down three equations in terms of x_1, x_2, and x_3 for the amount of money each has after three games.

(b) Solve the linear equations obtained above to determine x_1, x_2, and x_3.

Required Knowledge: **Gauss-Jordan elimination, row operations, reduced row echelon form, notions of lead/free variables.**

Solution:

1a: After the first round of game, the players will have

$$\begin{aligned}
&\text{Player 1:} \quad x_1 - (x_2 + x_3) = x_1 - x_2 - x_3\\
&\text{Player 2:} \quad x_2 + x_2 = 2x_2\\
&\text{Player 3:} \quad x_3 + x_3 = 2x_3
\end{aligned}$$

as Player 1 has lost the round. After the second round, they will have

$$\begin{aligned}
&\text{Player 1:} \quad (x_1 - x_2 - x_3) + (x_1 - x_2 - x_3) = 2x_1 - 2x_2 - 2x_3\\
&\text{Player 2:} \quad 2x_2 - (x_1 - x_2 - x_3) - 2x_3 = -x_1 + 3x_2 - x_3\\
&\text{Player 3:} \quad 2x_3 + 2x_3 = 4x_3
\end{aligned}$$

and after the third

$$\begin{aligned}
&\text{Player 1:} \quad (2x_1 - 2x_2 - 2x_3) + (2x_1 - 2x_2 - 2x_3) = 4x_1 - 4x_2 - 4x_3\\
&\text{Player 2:} \quad (-x_1 + 3x_2 - x_3) + (-x_1 + 3x_2 - x_3) = -2x_1 + 6x_2 - 2x_3\\
&\text{Player 3:} \quad 4x_3 - (2x_1 - 2x_2 - 2x_3) - (-x_1 + 3x_2 - x_3) = -x_1 - x_2 + 7x_3
\end{aligned}$$

as the losers are respectively, Player 2 and 3. As they all have €24 after three rounds, we arrive at the linear equation:

$$\begin{aligned}
4x_1 - 4x_2 - 4x_3 &= 24\\
-2x_1 + 6x_2 - 2x_3 &= 24\\
-x_1 - x_2 + 7x_3 &= 24.
\end{aligned}$$

1b: We first form the augmented matrix:

$$\begin{pmatrix}
-1 & -1 & 7 & \vdots & 24\\
-2 & 6 & -2 & \vdots & 24\\
4 & -4 & -4 & \vdots & 24
\end{pmatrix}$$

Then, we perform row operations to put the augmented matrix into the *reduced row echelon form*:

$$\begin{pmatrix} -1 & -1 & 7 & \vdots & 24 \\ -2 & 6 & -2 & \vdots & 24 \\ 4 & -4 & -4 & \vdots & 24 \end{pmatrix} \xrightarrow{\mathbf{1st}=-1\times\mathbf{1st}} \begin{pmatrix} 1 & 1 & -7 & \vdots & -24 \\ -2 & 6 & -2 & \vdots & 24 \\ 4 & -4 & -4 & \vdots & 24 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -7 & \vdots & -24 \\ -2 & 6 & -2 & \vdots & 24 \\ 4 & -4 & -4 & \vdots & 24 \end{pmatrix} \xrightarrow{\mathbf{2nd}=2\times\mathbf{1st}+\mathbf{2nd}} \begin{pmatrix} 1 & 1 & -7 & \vdots & -24 \\ 0 & 8 & -16 & \vdots & -24 \\ 4 & -4 & -4 & \vdots & 24 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -7 & \vdots & -24 \\ 0 & 8 & -16 & \vdots & -24 \\ 4 & -4 & -4 & \vdots & 24 \end{pmatrix} \xrightarrow{\mathbf{3rd}=-4\times\mathbf{1st}+\mathbf{3rd}} \begin{pmatrix} 1 & 1 & -7 & \vdots & -24 \\ 0 & 8 & -16 & \vdots & -24 \\ 0 & -8 & 24 & \vdots & 120 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -7 & \vdots & -24 \\ 0 & 8 & -16 & \vdots & -24 \\ 0 & -8 & 24 & \vdots & 120 \end{pmatrix} \xrightarrow{\mathbf{2nd}=\frac{1}{8}\times\mathbf{2nd}} \begin{pmatrix} 1 & 1 & -7 & \vdots & -24 \\ 0 & 1 & -2 & \vdots & -3 \\ 0 & -8 & 24 & \vdots & 120 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -7 & \vdots & -24 \\ 0 & 1 & -2 & \vdots & -3 \\ 0 & -8 & 24 & \vdots & 120 \end{pmatrix} \xrightarrow{\mathbf{3rd}=\frac{1}{8}\times\mathbf{3rd}} \begin{pmatrix} 1 & 1 & -7 & \vdots & -24 \\ 0 & 1 & -2 & \vdots & -3 \\ 0 & -1 & 3 & \vdots & 15 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -7 & \vdots & -24 \\ 0 & 1 & -2 & \vdots & -3 \\ 0 & -1 & 3 & \vdots & 15 \end{pmatrix} \xrightarrow{\mathbf{3rd}=\mathbf{2nd}+\mathbf{3rd}} \begin{pmatrix} 1 & 1 & -7 & \vdots & -24 \\ 0 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -7 & \vdots & -24 \\ 0 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 12 \end{pmatrix} \xrightarrow{\mathbf{2nd}=2\times\mathbf{3rd}+\mathbf{2nd}} \begin{pmatrix} 1 & 1 & -7 & \vdots & -24 \\ 0 & 1 & 0 & \vdots & 21 \\ 0 & 0 & 1 & \vdots & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -7 & \vdots & -24 \\ 0 & 1 & 0 & \vdots & 21 \\ 0 & 0 & 1 & \vdots & 12 \end{pmatrix} \xrightarrow{\mathbf{1st}=7\times\mathbf{3rd}+\mathbf{1st}} \begin{pmatrix} 1 & 1 & 0 & \vdots & 60 \\ 0 & 1 & 0 & \vdots & 21 \\ 0 & 0 & 1 & \vdots & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & \vdots & 60 \\ 0 & 1 & 0 & \vdots & 21 \\ 0 & 0 & 1 & \vdots & 12 \end{pmatrix} \xrightarrow{\mathbf{1st}=-1\times\mathbf{2nd}+\mathbf{1st}} \begin{pmatrix} 1 & 0 & 0 & \vdots & 39 \\ 0 & 1 & 0 & \vdots & 21 \\ 0 & 0 & 1 & \vdots & 12 \end{pmatrix}.$$

There are no free variables and hence the solution is unique: $x_1 = 39$, $x_2 = 21$, and $x_3 = 12$.

2 Matrix multiplication (2 + 18 = 20 pts)

The *trace* of a square matrix is the sum of its diagonal entries: For $M \in \mathbb{F}^{n\times n}$,

$$\operatorname{trace}(M) = \sum_{i=1}^{n} [M]_{ii}.$$

Let $A \in \mathbb{F}^{q\times r}$ and $B \in \mathbb{F}^{r\times q}$.

(a) What are the sizes of AB and BA?

(b) Show that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$.

REQUIRED KNOWLEDGE: **Definition of matrix multiplication.**

SOLUTION:

2a: According to the definition of matrix multiplication, we have $AB \in \mathbb{R}^{q\times q}$ and $BA \in \mathbb{R}^{r\times r}$

2b: Note that

$$\begin{aligned}\operatorname{trace}(AB) &= \sum_{i=1}^{q} [AB]_{ii} = \sum_{i=1}^{q} \sum_{k=1}^{r} [A]_{ik}[B]_{ki} = \sum_{k=1}^{r} \sum_{i=1}^{q} [A]_{ik}[B]_{ki} \\ &= \sum_{k=1}^{r} \sum_{i=1}^{q} [B]_{ki}[A]_{ik} = \sum_{k=1}^{r} [BA]_{kk} = \operatorname{trace}(BA)\end{aligned}$$

where the first equality follows from the definition of trace, the second from the definition of matrix multiplication, the third from the fact that the order of sums does not matter as long as there are only finitely many summands, the fourth from the commutativity of scalar multiplication, the fifth from the definition of matrix multiplication, and the last from the definition of trace.

3 Matrices and their properties (4 + 4 + 4 + 4 + 4 = 20 pts)

Let M be an 4×4 matrix with the characteristic polynomial $p_M(\lambda) = \lambda(\lambda - 1)(\lambda^2 + 1)$.

(a) Is M nonsingular? Justify your answer.

(b) Determine the eigenvalues of M.

(c) Determine the trace of M.

(d) Determine the determinant of M.

(e) Is M diagonalizable? Justify your answer.

REQUIRED KNOWLEDGE: **Eigenvalues, determinant, trace, diagonalizability.**

SOLUTION:

3a: The matrix M is singular as it has a zero eigenvalue.

3b: The eigenvalues are the roots of the characteristic polynomial. So, they are 0, 1, and $\pm i$.

3c: The trace of a matrix is equal to the sum of its eigenvalues. So, $\text{trace}(M) = 1$.
3d: The determinant of a matrix is equal to the product of its eigenvalues. So, $\det(M) = 0$.
3e: Since M has 4 distinct eigenvalues, it is diagonalizable.

4 Determinants and diagonalization (2 + 8 + 3 + 12 = 25 pts)

A *tridiagonal* matrix is a square matrix that has nonzero elements only on the main diagonal, the first diagonal below the main diagonal, and the first diagonal above the main diagonal. Consider the sequence of tridiagonal matrices $A_n \in \mathbb{R}^{n\times n}$ given by:

$$A_1 = 3,\ A_2 = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix},\ A_3 = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix},\ A_4 = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 2 & 3 \end{bmatrix},\ A_5 = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 3 \end{bmatrix}, \ldots$$

Let $d_n := \det(A_n)$.

(a) Verify that $d_1 = 3$ and $d_2 = 7$.

(b) By using cofactor expansion, determine the numbers p and q such that $d_k = pd_{k-1} + qd_{k-2}$ for all $k \geqslant 3$.

(c) Note that $\begin{bmatrix} d_k \\ d_{k-1} \end{bmatrix} = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_{k-1} \\ d_{k-2} \end{bmatrix}$ for all $k \geqslant 3$ where p and q are as obtained above. Let $M = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$. Show that $d_k = \begin{bmatrix} 1 & 0 \end{bmatrix} M^{k-2} \begin{bmatrix} d_2 \\ d_1 \end{bmatrix}$ for all $k \geqslant 3$.

(d) Determine a nonsingular matrix X and a diagonal matrix D such that $X^{-1}MX = D$. Determine M^k for all $k \geqslant 0$. Determine d_k for all $k \geqslant 3$.

Required Knowledge: **Cofactor expansion, effects of EROs on the determinant, diagonalization.**

Solution:

4a: By definition, we have

$$d_1 = \det(A_1) = \det(3) = 3 \qquad \text{and} \qquad d_2 = \det(A_2) = \det\left(\begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}\right) = 3\cdot 3 - 1\cdot 2 = 7.$$

4b: Note that

$$\begin{aligned} d_k &= \det(A_k) \\ &= 3\det(A_{k-1}) - \det\left(\left[\begin{array}{c:cccc} 2 & 1 & 0 & \cdots & 0 \\ \hdashline \mathbf{0}_{k-2} & & & A_{k-2} & \end{array}\right]\right) \\ &= 3d_{k-1} - 2\det(A_{k-2}) \\ &= 3d_{k-1} - 2d_{k-2}. \end{aligned}$$

for all $k \geqslant 3$. Therefore, $p = 3$ and $q = -2$.

4c: Note that

$$\begin{bmatrix} d_3 \\ d_2 \end{bmatrix} = M \begin{bmatrix} d_2 \\ d_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} d_4 \\ d_3 \end{bmatrix} = M \begin{bmatrix} d_3 \\ d_2 \end{bmatrix} = M^2 \begin{bmatrix} d_2 \\ d_1 \end{bmatrix}.$$

By repeating the same argument above, we obtain

$$\begin{bmatrix} d_k \\ d_{k-1} \end{bmatrix} = M^{k-2} \begin{bmatrix} d_2 \\ d_1 \end{bmatrix}$$

for all $k \geqslant 3$. By pre-multiplying by $\begin{bmatrix} 1 & 0 \end{bmatrix}$, we obtain

$$d_k = \begin{bmatrix} 1 & 0 \end{bmatrix} M^{k-2} \begin{bmatrix} d_2 \\ d_1 \end{bmatrix}.$$

4d: Note that

$$\det(M-\lambda I)=\det\left(\begin{bmatrix}3-\lambda & -2\\ 1 & -\lambda\end{bmatrix}\right)=\lambda^2-3\lambda+2=(\lambda-1)(\lambda-2).$$

Therefore, the eigenvalues of the matrix M are $\lambda_1=1$ and $\lambda_2=2$. To compute eigenvectors, we should solve

$$0=(M-\lambda_1 I)x_1=\begin{bmatrix}2 & -2\\ 1 & -1\end{bmatrix}x_1$$
$$0=(M-\lambda_2 I)x_2=\begin{bmatrix}1 & -2\\ 1 & -2\end{bmatrix}x_2.$$

These would yield, for instance,

$$x_1=\begin{bmatrix}1\\1\end{bmatrix}\qquad\text{and}\qquad x_2=\begin{bmatrix}2\\1\end{bmatrix}.$$

Let X be defined by

$$X:=\begin{bmatrix}x_1 & x_2\end{bmatrix}=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}.$$

Note that

$$X^{-1}=\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}$$

and also that

$$X^{-1}MX=\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}=D.$$

Since $M=XDX^{-1}$, we know that $M^k=XD^kX^{-1}$. Hence, we get

$$M^k=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}1^k & 0\\ 0 & 2^k\end{bmatrix}\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}=\begin{bmatrix}2^{k+1}-1 & 2-2^{k+1}\\ 2^k-1 & 2-2^k\end{bmatrix}$$

for all $k\geqslant 0$. Note that

$$\begin{aligned}
d_k &= \begin{bmatrix}1 & 0\end{bmatrix}M^{k-2}\begin{bmatrix}d_2\\ d_1\end{bmatrix}\\
&= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}2^{k-1}-1 & 2-2^{k-1}\\ 2^{k-2}-1 & 2-2^{k-2}\end{bmatrix}\begin{bmatrix}7\\ 3\end{bmatrix}\\
&= \begin{bmatrix}2^{k-1}-1 & 2-2^{k-1}\end{bmatrix}\begin{bmatrix}7\\ 3\end{bmatrix}\\
&= 7(2^{k-1}-1)+3(2-2^{k-1})\\
&= 4\cdot 2^{k-1}-1\\
&= 2^{k+1}-1.
\end{aligned}$$

for all $k\geqslant 3$. Since $d_1=3$ and $d_2=7$, we have $d_k=2^{k+1}-1$ for all $k\geqslant 1$.